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A CONTRIBUTION TO THE INTERPRETATION OF GEOMETRIC PASSAGE OF THE DIALOGUE MENON, PLATOS, (86e - 87b) *

Below is published in English the main part of a treatise, which is published in Greek in the «Platon» (Year Γ', part B', p. 218=227, 1951).

It is about the passage where the discovery of the possibility or not of the inscription of plane surface in the form of a triangle in a circle. In the issue of 1951 the two solutions of the problem by 1) Butcher — Heath and 2) A. S. L. Farquarson [1) T. L. Heath, A history of Greek Mathematics I p, 229—301, 1921. 2) Classical Quartarly XVII, 1, Januar 1923] are also mentioned in detail.

The solution of this problem I propose is the following :

Hypothesis I (fig. 1)

Suppose that the figure on the ground, in the above passage of the Meno is the square EDGO and that its doubled square is EGHZ. We fill out the square ABCD ; let AEHB be the given rectilinear area that must be inscribed as a triangle in the given circle with center O.

We extend the rectangle AEHB along the given line AD(=AB), from which the rectangle of equal area EDCH is left **. As is obvious from the figure, the rectangle AEHB cannot be inscribed as a triangle in the given circle, because it consists of two squares, AEOZ and ZOHB ; for these squares, according to the solution of the previous geometric passage of the dialogue (81e—85c), are equal to the square EGHZ, which, as is

* Ἡ παροῦσα μελέτη εἰς τὴν ἀγγλικὴν γλῶσσαν ἀποτελεῖ τὸ κύριον μέρος πραγματείας ἡμῶν δημοσιευθείσης ἑλληνιστὶ εἰς τὸν Πλάτωνα (Ἔτος Γ' τεῦχος Β', σελὶς 218—227, 1951). Πρόκειται περὶ τοῦ χωρίου, ὅπου συζητεῖται ἡ εὕρεσις τοῦ δυνατοῦ ἢ μὴ τῆς ἐγγραφῆς ἐπιπέδου εὐθυγράμμου ἐπιφανείας ὑπὸ μορφὴν τριγώνου εἰς κύκλον.

Εἰς τὸ τὸ τεῦχος τοῦ 1951 ἐκτίθενται λεπτομερῶς καὶ δύο λύσεις τοῦ προβλήματος ὑπὸ 1) Butcher - Heath καὶ 2) A. S. L. Farquarson [1) I. L. Heath, A history of Greek Mathematics, I, pp. 229—301, 1921, 2) A. S. L. Farquarson . . . , Classical Quartarly XVII. I Januar 1923].

** This view is based on the 27th theorem of the vi book of Euclid's Elements.

known, is the maximum of the parallelograms that can be inscribed in the circle.

Hypothesis II (*fig.* 1)

We leave out half of the rectangle AEHB, i.e. the square ZOHB, We extend the remaining square AEOZ along the given line AD of the figure. Thus, EDGO, which is equal to the extended AEOZ, is left out from ADGZ, which is equal to AEHB. It is obvious that the square AEOZ can be inscribed in the given circle as a triangle EGZ, the base of which,

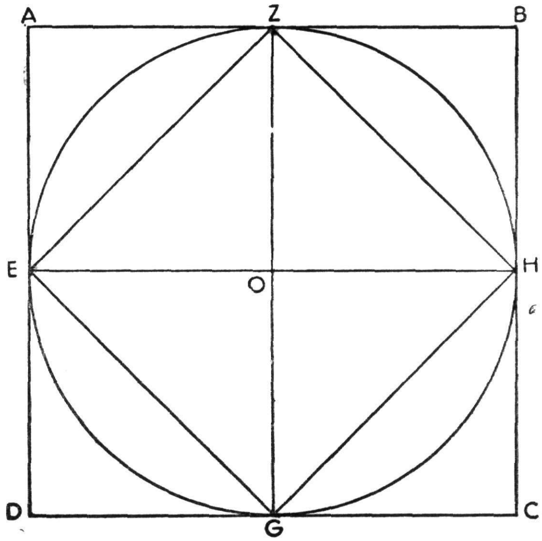


fig. 1.

GZ, is equal to AD (i.e. the diameter of the circle). Therefore, we can consider the longest of the lines of the triangle which is equal to the diameter as the given fixed side of the area and proceed with various transformations. It is also obvious that if we keep AB fixed and move ZO along a line parallel to itself in the direction of AE, we obtain areas progressively smaller than the square AEOZ, each of which can be inscribed in the given circle.

Hypothesis III (fig. 2)

We suppose that at the time the M e n o was written it was known that the largest triangle that can be inscribed in a circle is an equilateral the area of which is equal to $\frac{3}{4} r^2 \sqrt{3}$, where r is the radius of the circle.

If the given area is larger than the square AEOZ, then the maximum area which can be inscribed as a triangle in the given circle (if transformed to a rectangle with AE, the radius of the circle, as the constant side) will have the side $AB = \frac{3}{4} r \sqrt{3}$ as its other side [Note: We can draw the straight line $r \sqrt{3}$ with a rule and compass as follows: We suppose a right triangle ABC, of which A is the right angle and AB=AC=r its perpendicular side. At one of the ends of the hypotenuse, e.g. B, we draw a perpendicular line and take a part of it so that BD=r. We draw a line from D to C. The hypotenuse CD of the new right triangle BDC (where B is the right angle) equals $r \sqrt{3}$]. From a point B we draw a perpendicular to the diameter of the circle and extend it to a point P. Thus, we have

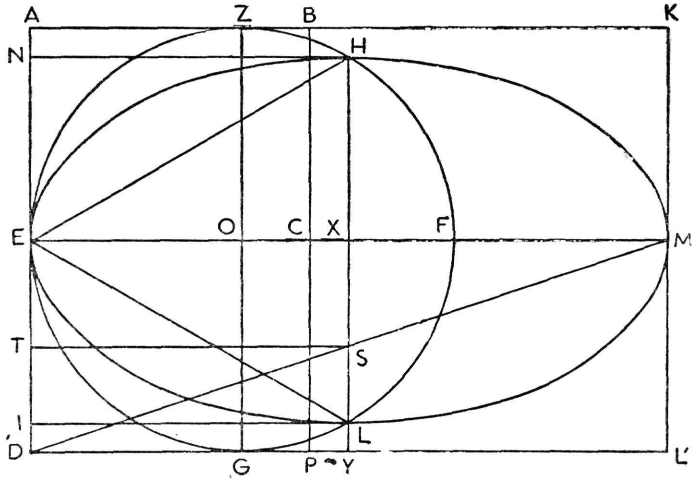


fig. 2

extended the rectangle AECB along the given line AD leaving out the rectangle EDPC, which is equal to AECB.

In our opinion, the problem is now how to inscribe the area AECB in the given circle as a triangle. There are several methods to solve this problem. First, we may inscribe an hexagon in the circle and join every two vertexes together. Second, we may construct the hyperbole

$$xy = b^2 = \frac{3}{4} r^2 \sqrt{3}$$

(where x=AB, y=AE and b^2 the area of the given surface). If we draw a perpendicular on the point of contact and prolong the line until it cuts the circumference, we obtain the base of the desired equilateral triangle with vertex point E.

In this case, we suppose that we can obtain the desired equilateral triangle through an ellipsis, on the basis of the words ἐλλείπειν κλπ. of the text as well as Theorem 28 of book VI of Euclid's Elements and Theorem 13 of book I of Apollonius' Conics.

Analysis.

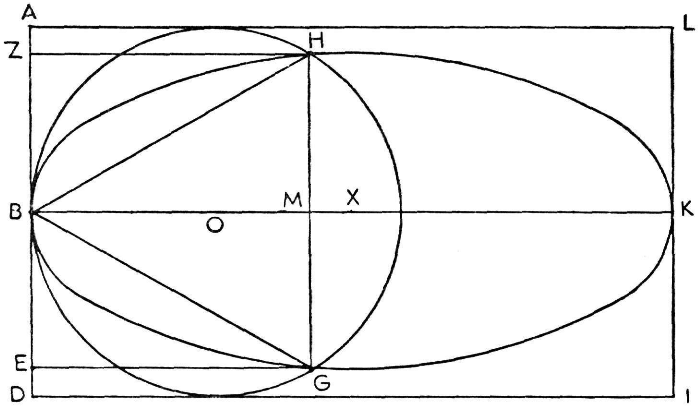
We take a point H in the circumference of a circle (fig. 2) so that the parallelogram NEXH is equivalent to the parallelogram AECB. It is obvious that the triangle EHL is equivalent to the parallelogram NEXH=AECB. $EX=\frac{3r}{2}$.

Synthesis.

With $\alpha=EX=\frac{3r}{2}$ as the semiaxis of the ellipsis and with the radius of the circle $r=2p=ED$ we construct an ellipsis, in accordance with Theorem 28 of book VI of Euclid's Elements and Theorem 13 of book I of Apollonius' Conics. This construction takes place as follows: We form the rectangle EDL′M (fig. 2), in which the great axis EM=2a and the parameter of the ellipsis ED=r=2p. We draw the diagonal MD. From a point X of the axis EM we draw the line XY perpendicular to DL′. This line sects the diagonal MD at point S. We then prolong the parallelogram ETSX along the line ED. From this the rectangle TDYS is left out, which is equal to ETSX as well as similar to EDL′M and situated similarly to the latter. The area of the parallelogram ETSX is equivalent to a square with side the line XL. The point L is a point of the ellipsis. In the same way, if we draw the symmetrical diagonal MA of the parallelogram AEMK we obtain the point H of the ellipsis. These two points L and H determine the axis b of the ellipsis, which is the longest perpendicular that can be drawn to the great axis according to the definition given in Theor. 27 of book VI of Euclid's Elements. With a similar construction, i. e. by drawing perpendiculars from whatever two points of the axis EM to DL′, we obtain the points in which they sect the diagonal DM ; from these we draw parallel lines to EM (which are perpendicular to ED). The parallelograms on the side of the small axis (with the exception of those below the parallel drawn from the diagonal) if transformed into squares give the points of the ellipsis. It is clear that each of the parallelograms thus transformed into squares is smaller than ETSX, according to Theor. 27 of book VI of Euclid's Elements. The section of the ellipsis and of the circle give the two points on the circumference of the circle which joined together form the basis of the inscribed triangle which has point E as vertex and which is the maximum triangle that can be inscribed in the circle.

Hypothesis IV.

If the side of the surface which is transformed into a rectangle is longer than the radius of the circle and indeed longer that $\frac{3}{4}$ r $\sqrt{3}$ then



feg. 3

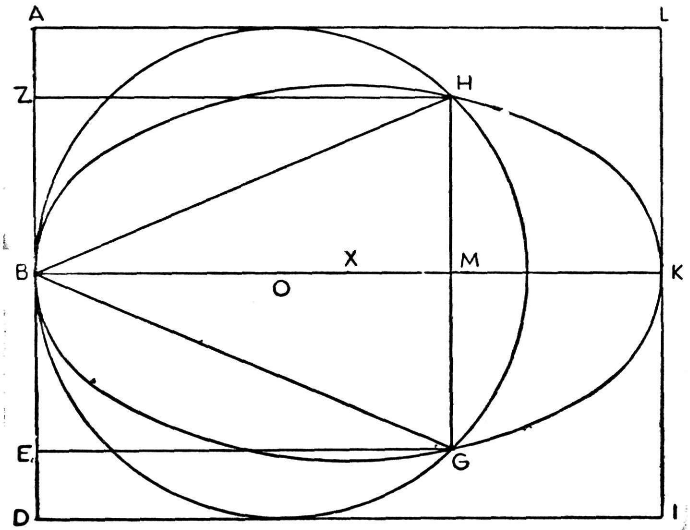


fig 4

the given surface is larger than the maximum rectangle, i.e.r $\frac{3}{4}$ r $\sqrt{3}$ (which can be inscribed in the circle as triangle) and therefore cannot be inscribed in the given circle as triangle. In our opinion this is the case referred to in the dialogue with the words «εἰ ἀδύνατόν ἐστι ταῦτα παθεῖν»

i. e. if it is impossible «τὸ ἐλλείπειν οἷον ἂν αὐτὸ τὸ παρατεταμένον ᾖ».

In a modern formulation, hypothesis 3 can be solved with the equation $y^2=2px-\frac{px^2}{a}$ and the equation of the circle $y^2=2rx-x^2$, where $2p=r$, $a=$ the great semiaxis and $b^2\leqq\frac{3}{4}r^2\sqrt{3}$ is the area of the given surface.

In fig. 3 obtain the inscribed triangle $b^2<\frac{3}{4}r^2\sqrt{3}$, where $a=\frac{3r}{2}+h$, while in fig. 4 we obtain the triangle $\gamma^2<\frac{3}{4}r^2\sqrt{3}$, where $a=\frac{3r}{2}-h$ $\left(o<h<\frac{r}{2}\right)$ and $2p=r$. The ellipsis and the circle ought to be cotangental at the origin of the coordinates,